General soliton solutions of an n-dimensional nonlinear Schrodinger equation

1992 J. Phys. A: Math. Gen. 25 L515

(http://iopscience.iop.org/0305-4470/25/9/002)

J. Phys. A: Math. Gen. 25 (1992) L515–L519. Printed in the UK

LETTER TO THE EDITOR

General soliton solutions of an n-dimensional nonlinear Schrödinger equation

Wenhua Hai

Journal of Hunan Educational Institute, Changsha 410012, Hunan, People's Republic of China

Received 17 September 1991, in final form 5 January 1992

Abstract. Applying the method of functions transformation, an n-dimensional nonlinear Schrödinger equation is changed into a sine-Gordon equation which depends only on one function ξ. The general solution of the equations of ξ leads to a general soliton solution of NLS. It contains some interesting specific solutions, such as the N multiple solitons, the propagational breathers and the quadric solitons. Their properties are simply discussed.

The nonlinear Schrödinger equation NLS is an important equation in physics. For the one-dimensional case, many results on the equation have been given in a number of articles [1–3]. Published works are fewer for the multidimensional NLS, because they contain some difficult problems. Although it is known that the multidimensional solutions of NLS are unstable [4, 5], solution to the n-dimensional NLS is still of interest [6, 7]. In particular, these n-dimensional solutions contain some stable plane soliton solutions.

Consider an n-dimensional NLS

$$\mathrm{i}\partial_0\psi + \partial_i\partial_i\psi = a\psi^2\psi^* - b\psi \tag{1}$$

where $\partial_0 = \partial/\partial x_0 = \partial/\partial t$, $\partial_i = \partial/\partial x_i$, a, b = constants, ψ^* denotes the complex conjugate function of ψ. Here, and throughout, we use the summation convention: a greek index runs from 0 to $n-1$ and any other index runs from 1 to $n-1$, unless it is particularly stated otherwise. We come to find the solutions of (1) in the form

$$\psi = u(x)\,\mathrm{e}^{\mathrm{i}c_\alpha x_\alpha} \qquad c_\alpha = c_\alpha^* = \text{constant} \qquad u(x) = u^*(x). \tag{2}$$

By inserting (2) into (1), one obtains

$$\mathrm{i}\partial_0 u + \mathrm{i}2c_i\partial_i u + \partial_i\partial_i u = au^3 - cu \qquad c = b - c_0 - c_ic_i. \tag{3}$$

Let us make a function transformation

$$u = \sqrt{\frac{c}{a}}\sin\frac{\varphi}{2}. \tag{4}$$

We then have

$$\begin{aligned}
&\mathrm{i}\partial_0 u + \mathrm{i}2c_i\partial_i u = \frac{\mathrm{i}}{2}\sqrt{\frac{c}{a}}\cos\frac{\varphi}{2}(\partial_0\varphi + 2c_i\partial_i\varphi)\\
&\partial_i\partial_i u = \frac{1}{2}\sqrt{\frac{c}{a}}\cos\frac{\varphi}{2}\left\langle\partial_i\partial_i\varphi - \frac{1}{2}\operatorname{tg}\frac{\varphi}{2}\,\partial_i\varphi\partial_i\varphi\right\rangle\\
&a^3 - cu = -\frac{c}{2}\sqrt{\frac{c}{a}}\cos\frac{\varphi}{2}\sin\varphi.
\end{aligned} \tag{5}$$

Given (5), (3) becomes

$$\mathrm{i}\partial_0\varphi + \mathrm{i}2c_i\partial_i\varphi + \partial_i\partial_i\varphi - \tfrac{1}{2}\,\mathrm{tg}\,\frac{\varphi}{2}\,\partial_i\varphi\partial_i\varphi = -c\sin\varphi. \tag{6}$$

Setting $\varphi = \varphi(\xi)$ which is a function of another function ξ only, we easily see that

$$\begin{aligned} &\partial_\alpha\varphi = \partial_\alpha\xi\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} \\ &\partial_i\varphi\partial_i\varphi = \partial_i\xi\partial_i\xi\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}\right)^2 \\ &\partial_i\partial_i\varphi = \partial_i\xi\partial_i\xi\frac{\mathrm{d}^2\varphi}{\mathrm{d}\xi^2} + \partial_i\partial_i\xi\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}. \end{aligned} \tag{7}$$

Substituting (7) into (6) yields

$$(\mathrm{i}\partial_0\xi + \mathrm{i}2c_i\partial_i\xi + \partial_i\partial_i\xi)\,\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} + \partial_i\xi\partial_i\xi\left[\frac{\mathrm{d}^2\varphi}{\mathrm{d}\xi^2} - \frac{1}{2}\,\mathrm{tg}\,\frac{\varphi}{2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}\xi}\right)^2\right] = -c\sin\varphi. \tag{8}$$

Explicitly, some solutions of (8) obey the following system of equations:

$$\mathrm{i}\partial_0\xi + \mathrm{i}2c_i\partial_i\xi = \partial_i\partial_i\xi = 0 \qquad \partial_i\xi\partial_i\xi = 1 \tag{9}$$

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = \sqrt{2c}\cos\frac{\varphi}{2} \qquad \frac{\mathrm{d}^2\varphi}{\mathrm{d}\xi^2} = -\frac{c}{2}\sin\varphi. \tag{10}$$

Equation (10) is equivalent to a sine-Gordon equation, its solution is a well known soliton

$$\varphi = 4\,\mathrm{tg}^{-1}\exp(\sqrt{2c}\,\xi + \xi_0) - \pi \qquad \xi_0 = \text{constant} \tag{11}$$

Applying (11) and (4) to (2), we obtain the soliton solution of NLS equation (1) in the form

$$\psi = -\sqrt{\frac{c}{a}}\cos[2\,\mathrm{tg}^{-1}\exp(\sqrt{2c}\,\xi + \xi_0)]\exp(\mathrm{i}c_\alpha x_\alpha) \tag{12}$$

where ξ denotes a solution of (9). Because (9) has many different solutions, (12) includes many interesting solitons of NLS.

Now we come to find a kind of general solution of (9) in the form

$$\xi = F(\zeta_j) + d_\alpha x_\alpha \qquad \zeta_j = b_{j\alpha}x_\alpha + \varepsilon_j \qquad d_\alpha,\, b_{j\alpha},\, \varepsilon_j = \text{constant} \tag{13}$$

where $F(\zeta_j)$ denotes an arbitrary function of ζ_j. Combining (9) and (13) we easily obtain

$$\begin{aligned} &\mathrm{i}\partial_0\xi + 2\mathrm{i}c_i\partial_i\xi = (\mathrm{i}b_{j0} + 2\mathrm{i}c_ib_{ji})\,\frac{\partial F}{\partial\zeta_j} + d_0 + 2\mathrm{i}c_id_i = 0 \\ &\partial_i\xi\partial_i\xi = b_{ji}b_{ki}\frac{\partial F}{\partial\zeta_j}\frac{\partial F}{\partial\zeta_k} + 2d_ib_{ji}\frac{\partial F}{\partial\zeta_j} + d_id_i = 1 \\ &\partial_i\partial_i\xi = b_{ji}b_{ki}\frac{\partial^2 F}{\partial\zeta_j\partial\zeta_k} = 0. \end{aligned} \tag{14}$$

$F(\zeta_j)$ to be arbitrary leads to the conditions

$$b_{j0} + 2c_ib_{ji} = 0 \qquad d_0 + 2\mathrm{i}c_id_i = 0 \qquad d_ib_{ji} = 0 \qquad b_{ji}b_{ki} = 0 \qquad d_id_i = 1. \tag{15}$$

If we take $F(\zeta_j)$ for $j=1, 2, \ldots, N$, then (15) implies $(2+(N/2))(1+N)$ equations with $[(N+1)\cdot(n-1)+n]$ constants c_i, d_α, b_{ji} for $i=1, 2, \ldots, n-1$, $\alpha=0, 1, \ldots, n-1$, $j=1, 2, \ldots, N$. Therefore we require that N satisfies the inequality

$$\left(2+\frac{N}{2}\right)(1+N)\leqslant(N+1)\cdot(n-1)+n. \tag{16}$$

(16) and (13) show that there are three cases:

(1) $n\leqslant 2$, $N=0$, $\xi=d_\alpha x_\alpha$ denotes a plane solution;

(2) $n=3$, $N\leqslant 2$, ξ is a general solution with $N\leqslant 2$ variables;

(3) $n=4$, $N\leqslant 3$, ξ is a general solution with $N\leqslant 3$ variables.

The general solution (13) makes (12) and (11) a form of general soliton solution of NLS and sG equations. They contain some interesting specific solutions, such as the plane solitons, the N multiple solitons, the propagational breathers and the quadric solitons. We will discuss them respectively as the following.

(*a*) *The plane solitons.* This is a simple case. By taking all of b_{ji} in (15) to be equal to zero, $F(\zeta_j)$ becomes a constant so that

$$\xi=d_\alpha x_\alpha+\text{constant}.$$

In this case, (12) denotes a hyperplane soliton solution of NLS. This solution is stable, since the ξ is equivalent to a variable on one-dimensional space [4].

(*b*) *The N multiple soliton solutions.* Let us select the function $F(\zeta_j)$ in the form

$$F(\zeta_j)=\ln\sum_{j=1}^{N}\mathrm{e}^{\zeta_j} \tag{17}$$

where N denotes an arbitrary integral number satisfying (16). Application of (13) and (17) leads to

$$\begin{aligned}\xi&=F(\zeta_j)+d_\alpha x_\alpha=\ln\left(\mathrm{e}^{d_\alpha x_\alpha}\sum_{j=1}^{N}\mathrm{e}^{\zeta_j}\right)\\&=\ln\sum_{j=1}^{N}\exp[(d_\alpha+b_{j\alpha})x_\alpha+\varepsilon_j].\end{aligned} \tag{18}$$

Setting $a_{j\alpha}=d_\alpha+b_{j\alpha}$, the conditions (15) are simplified to

$$a_{j0}+2c_i a_{ji}=0 \qquad a_{ji}a_{ki}=1. \tag{19}$$

By direct calculation, we can easily prove that (18) is a solution of (9). Taking N equal to 1, 2, $\ldots$, N respectively, (18) gives N multiple wave solutions and (12) becomes N multiple soliton solutions of NLS (1). They have some well known properties which are similar to those of some solutions of wave equations [8–10].

(*c*) *The propagational breathers.* Considering $N=2$ in (18), we obtain

$$\xi=\ln(\mathrm{e}^{a_{1\alpha}x_\alpha+\varepsilon_1}+\mathrm{e}^{a_{2\alpha}x_\alpha+\varepsilon_2}) \tag{20}$$

which corresponds to a 2-soliton of NLS. For the four-dimensional case, we set

$$\begin{aligned}&x_0=t \qquad x_1=x \qquad x_2=y \qquad x_3=z\\&a_{10}=-a_{20} \qquad a_{11}=-a_{21} \qquad a_{12}=a_{22} \qquad a_{13}=a_{23}\\&\varepsilon_1=\ln A \qquad \varepsilon_2=\ln A+\mathrm{i}\pi\end{aligned} \tag{21}$$

then (20) becomes

$$\xi = \ln[A\, e^{a_{12}y+a_{13}z}(e^{a_{10}t+a_{11}x} - e^{-a_{10}t-a_{11}x})]$$
$$= \ln[A\, e^{a_{12}y+a_{13}z} \sinh(a_{10}t + a_{11}x)] \tag{22}$$

and (19) gives the conditions

$$a_{11}^2 + a_{12}^2 + a_{13}^2 = 1 \qquad a_{10} + 2c_1 a_{11} = 0 \qquad c_2 a_{12} + c_3 a_{13} = 0. \tag{23}$$

If a_{10}, a_{11} and A are some imaginary numbers such that

$$a_{10} = \mathrm{i}a_0 \qquad a_{11} = \mathrm{i}a_1 \qquad A = -\mathrm{i}B \tag{24}$$

then from (22) we have

$$\xi = \ln[B\, e^{a_{12}y+a_{13}z} \sin(a_0 t + a_1 x)]. \tag{25}$$

Inserting (25) into (12) or (11) yields a multidimensional breather which propagates in the x direction and has same value on the right line $a_{12}y + a_{13}z =$ constant. This is an interesting result.

(*d*) *The quadric solitons.* We know that any concrete soliton is a four-dimensional soliton with space shape. Its shape should be one of curved surfaces. We will show that the general solution (12) and (11) contain some quadric soliton solutions of NLS and SG.

In order to achieve the goal, we take the solution (13) in the form

$$\xi = \zeta_j \zeta_j + d_\alpha x_\alpha = b_{j\alpha} b_{j\beta} x_\alpha x_\beta + (2\varepsilon_j b_{j\alpha} + d_\alpha) x_\alpha + \varepsilon_j \varepsilon_j \qquad j = 1, 2, \ldots, N. \tag{26}$$

For the four-dimensional case, (26) implies

$$\begin{aligned}\xi = {} & b_{j1}b_{j1}x^2 + b_{j2}b_{j2}y^2 + b_{j3}b_{j3}z^2 + 2b_{j1}b_{j2}xy + 2b_{j2}b_{j3}yz + 2b_{j1}b_{j3}xz \\ & + (2\varepsilon_j b_{j1} + 2b_{j0}b_{j1}t + d_1)x + (2\varepsilon_j b_{j2} + 2b_{j0}b_{j2}t + d_2)y + (2\varepsilon_j b_{j3} + b_{j0}b_{j3}t + d_3)z \\ & + b_{j0}b_{j0}t^2 + (2\varepsilon_j b_{j0} + d_0)t + \varepsilon_j\varepsilon_j \qquad j = 1, 2, \ldots, N.\end{aligned} \tag{27}$$

It describes some general quadratic surfaces at any definite time. These quadrics include all of specific ones such as:

(i) *The sphere.* By choosing the constants of (27) as

$$b_{ji} b_{jk} = \delta_{ik} \qquad j = 1, 2, \ldots, N \tag{28}$$

we obtain a sphere with radius

$$R = \left[\left(\varepsilon_j b_{ji} + b_{j0} b_{j1} t + \frac{d_i}{2}\right)\left(\varepsilon_k b_{ki} + b_{k0} b_{ki} t + \frac{d_i}{2}\right) - b_{j0} b_{j0} t^2 - (2\varepsilon_j b_{j0} + d_0)t - \varepsilon_j \varepsilon_j\right]^{1/2} \tag{29}$$

which satisfies $R^2 > 0$, and with centre

$$G(-2\varepsilon_j b_{j1} - 2b_{j0}b_{j1}t - d_1,\ -2\varepsilon_j b_{j2} - 2b_{j0}b_{j2}t - d_2,\ -2\varepsilon_j b_{j3} - 2b_{j0}b_{j3}t - d_3) \tag{30}$$

which moves along a space line. Thus (27), (11) and (12) give some sphere solitons of NLS and SG.

(ii) *The ellipsoid.* Under the conditions

$$\begin{aligned}& b_{j1}b_{j1} > 0 \qquad b_{j2}b_{j2} > 0 \qquad b_{j3}b_{j3} > 0 \\ & b_{ji}b_{jk} = 0 \qquad i \neq k, \qquad j = 1, \ldots, N\end{aligned} \tag{31}$$

(27) become ellipsoids and (11), (12) the ellipsoid solitons.

(iii) *The hyperboloid.* Let the constants satisfy

$$b_{j1}b_{j1} > 0 \qquad b_{j2}b_{j2} > 0 \qquad b_{j3}b_{j3} < 0$$
$$b_{ji}b_{jk} = 0 \qquad i \neq k \qquad j = 1, \ldots, N. \tag{32}$$

From (27) we obtain some hyperboloids and (11), (12) give the corresponding hyperboloid solitons of NLS and SG.

(*e*) *The N multiple quadric solitons and the quadric breathers.* Further we select a specific solution as

$$\begin{aligned}\xi &= \ln \sum_{j=1}^{N} \exp \zeta_j^2 + d_\alpha x_\alpha \\ &= \ln\left[e^{d_\alpha x_\alpha} \sum_{j=1}^{N} \exp(b_{j\alpha}x_\alpha + \varepsilon_j)^2 \right] \\ &= \ln \sum_{j=1}^{N} \exp[(b_{j\alpha}x_\alpha + \varepsilon_j)^2 + d_\alpha x_\alpha].\end{aligned} \tag{33}$$

According to the discussions on (18), (25) and (26), it is clear that inserting (33) into (12) and (11), respectively, will give the N multiple quadric solitons and the quadric breathers of NLS and SG. Their properties are obvious and interesting. For example, the quadric breather behaves like a 'breathing abdomen'.

Lastly, we assert that (11), (12) and (13) can lead to many other new soliton solutions of NLS and SG. They will be quite complicated and interesting. All of these soliton and breather solutions can describe various important physical phenomena.

References

[1] Bullough R K and Caudrey P J 1980 *Solitons* (Berlin: Springer)
[2] Ginibre G J and Velo G 1979 *J. Funct. Anal.* **32**
[3] Chen Z Y, Huang N N and Xiao Y 1988 *Phys. Rev.* A **38** 4355
[4] Derrick G H 1964 *J. Math. Phys.* **5** 1252
[5] Hoburt R H 1965 *Proc. Phys. Soc.* **83** 610
[6] Ablowitz M J 1978 *Stud. Appl. Math.* **58** 17
[7] Makhankov V G 1978 *Phys. Rep.* **35** 1
[8] Gibbon J D, Freeman N C and Johnson R S 1978 *Phys. Lett.* **65A** 38
[9] Gibbon J D, Freeman N C and Davey A 1978 *J. Phys. A: Math. Gen.* **11** L93
[10] Lou S, Huang G and Ni G 1990 *Acta Phys. Sinica* **39** 1363